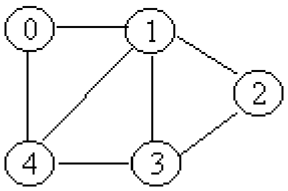
Graphs

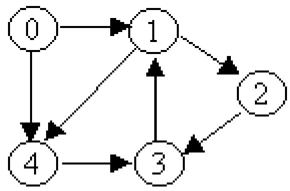
1. Basic definitions

- Graph G = (V, E) where V is a set of **vertices** and E is a set of **edges**. Each edge $e \in E$ is a 2-tuple of the form (v, w) where $v, w \in V$, and e is called an **incident** on v and w.

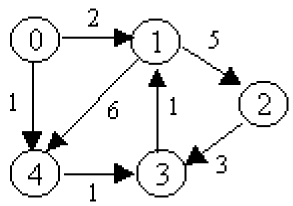


V = {0, 1, 2, 3, 4}
E = {(0, 1), (1, 2), (1, 3), (1, 4),
(2, 3), (3, 4), (4, 0)}

- An edge may be directed or undirected.
- An edge may also have a **weight**.

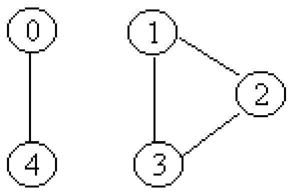


Directed, unweighted



Directed, weighted

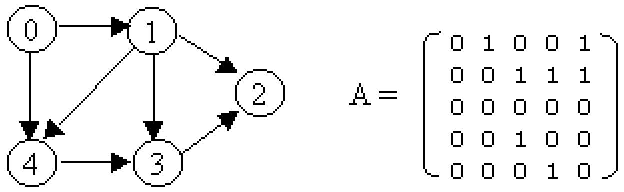
- A **path** is a sequence of vertices connected by edges, and represented as a sequence in 2 ways:
  - (v_0, e_1, v_1, e_2, v_2,..,v_{n-1}, e_n, v_n) -- alternating vertices and edges
  - (v_0, v_1, v_2,..,v_{n-1}, v_n) -- vertices only
- A graph is **connected** if, for any vertices v and w, there is a path from v to w.

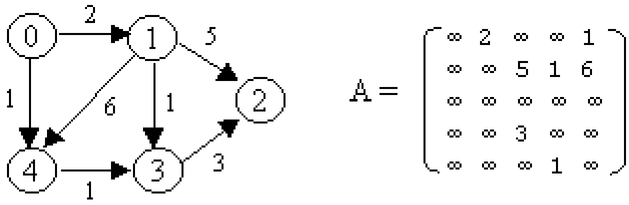


An unconnected graph

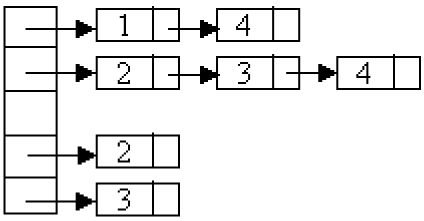
2. Representing Graphs

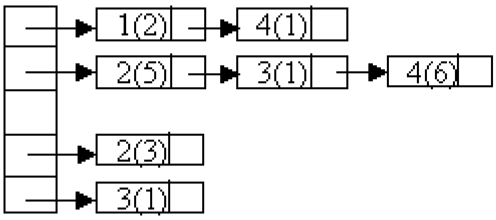
- Adjacency matrix
  - n by n matrix, where n is number of vertices
  - A(m,n) = 1 iff (m,n) is an edge, or 0 otherwise
  - For weighted graph: A(m,n) = w (weight of edge), or positive infinity otherwise





- Adjacency list
  - Each vertex has a linked list of edges
  - Edge stores destination and label
  - Better when adjacency matrix is **sparse**



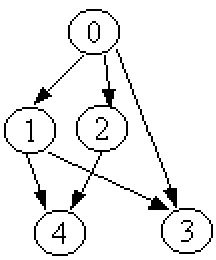


3. Graph Traversal

- Walk through a graph systematically in a predefined order -- Depth-first, or Breadth-first.

3.1 Depth-First Traversal

- Follow a path until it ends, or until a cycle. Use a <u>**stack**</u>.



Start vertex = 0
Assume vertex with smaller label is visited first.

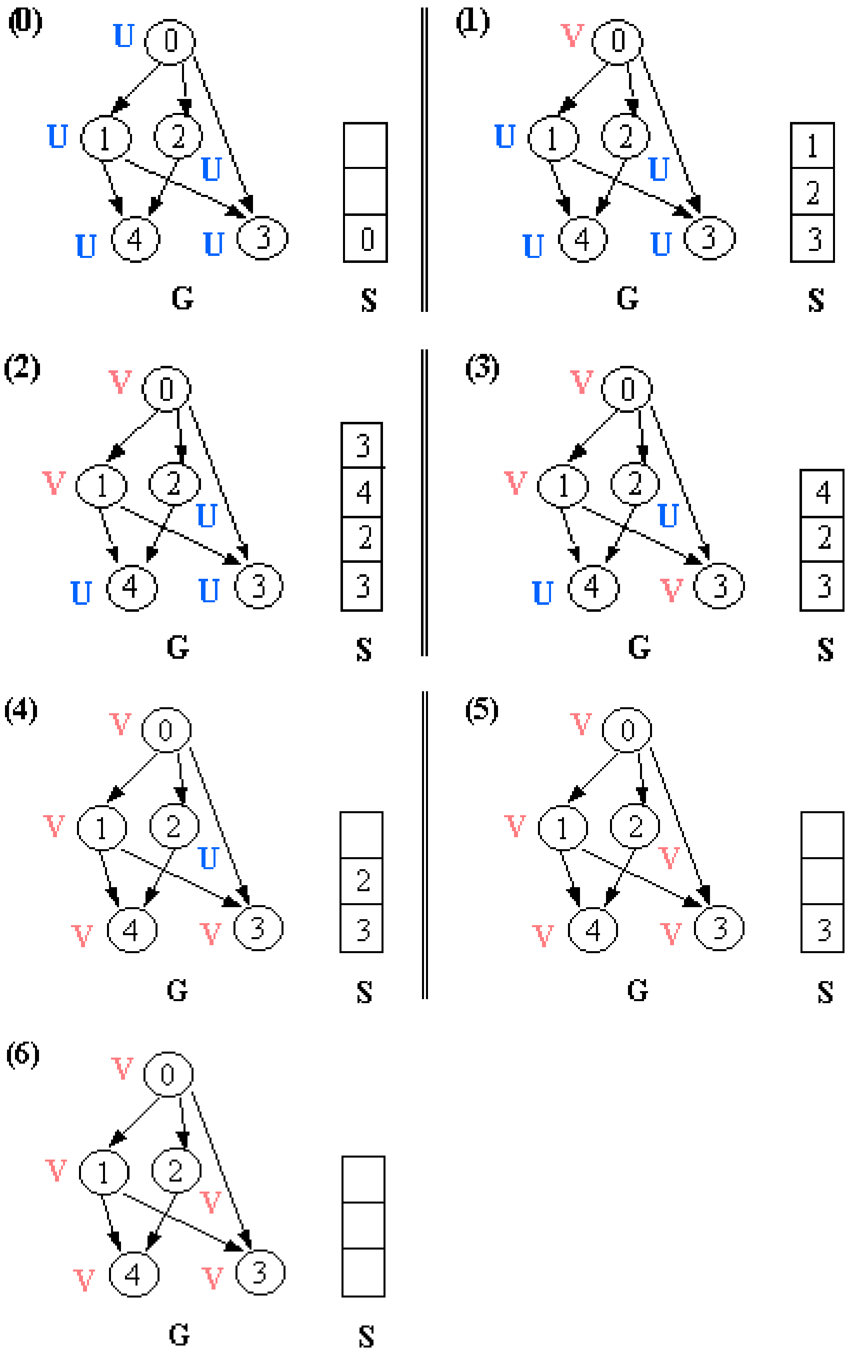
* Depth-first: 0, 1, 3, 4, 2

- <u>Algorithm</u>:

  Let G = (V, E) is a graph which is represented by an adjacency matrix Adj. Assume that nodes in a graph record visited/unvisited information.

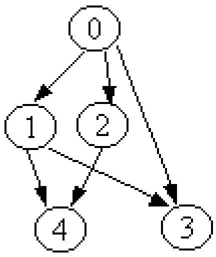
```
procedure DEPTH-FIRST (G)
1. Initialize all vertices as "unvisited".
2. Let S be a stack.
3. Push the root on S.
```

```
While S not empty, do
  begin
  Let n <- Pop S.
  If n is marked as "unvisited", then
    begin
    Mark n as "visited", and output n to the terminal.
     For each vertex v in Adj[n], do
       If v is marked as "unvisited", then // this test is actually redundant
         push v on S.
    end
  end
```



3.2 Breadth-First Traversal

- Visit nodes layer-by-layer. Use a **queue**.



Start vertex = 0
Assume vertex with smaller label is visited first.

- Breadth-first: 0, 1, 2, 3, 4

- Algorithm :

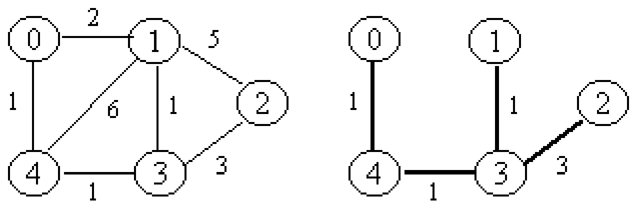
```
procedure BREADTH-FIRST (G)
Initialize all vertices as "unvisited".
Let Q be a queue.
Enqueue the root on Q.
While Q not empty, do
  begin
  n <- Dequeue Q.
  If n is marked as "unvisited", then
    begin
    Mark n as "visited", and output n to the terminal.
     For each vertex v in Adj[n], do
       If v is marked "unvisited", then
         enqueue v on Q.



    end
  end
```

4. Graph Search

- Two search methods corresponding to the two traversal schemes above: Depth-First Search (DFS) and Breadth-First Search (BFS).
- Terminate search/traversal as soon as the item is found.

5. Minimum Spanning Trees (MST)

- A minimum spanning tree T of an **undirected** graph G is a subgraph of G that **connects all the vertices** in G at the **lowest total cost**.



Graph G

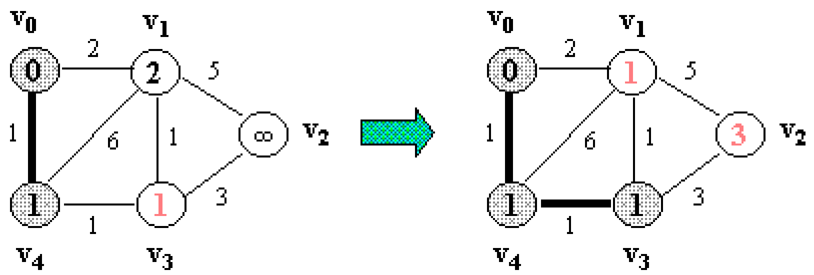
A Minimum Spanning Tree for G
(total cost = 6)

- MST is used as one of the most important tools to analyze **computer networks** (e.g. cabling, network load capacity, optimal flow).
- Two algorithms: **Prim's** algorithm and **Kruskal's** algorithm.
- They are both *greedy algorithms*.

6.1 Prim's Algorithm

- Maintains ONE TREE throughout the algorithm, and make it grow by adding edge by edge.
- The idea is to select the next edge

- which is adjacent from any vertex/node in the tree built so far; and
- which has the lowest weight among alternatives (i.e., all edges connected from any vertex/node in the tree built so far).



- Algorithm:

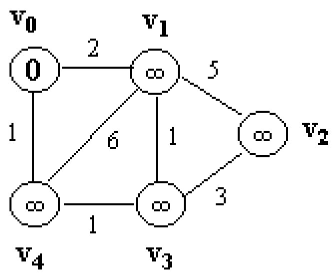
  Let G = (V, E) which is represented by an adjacency list **Adj**. Some support data structures:

  - **d** is an array of size |V|. Each d[i] contains the shortest distance for vertex i
  - **Q** is a **priority queue** of UNKNOWN vertices.
  - **p** is an array of size |V|. Each P[i] contains the **parent** of vertex i.
  - **s** is the source vertex.

```
PRIM(G, s)
1. Initialize d[s] with 0, P[s] with 0, and
    all other d[i] (i!=s) with a positive infinity and
              p[i] (i!=s) with 0.
2. Q <- V  // initialize Q with all vertices as UNKNOWN
3. while Q not empty do
4.   begin
5.   u <- ExtractMin(Q)  // Q is modified
6.   Mark u as KNOWN     // Dequeing u is the same as marking it as KNOWN
7.   for each vertex v in Adj[u] do
8.     begin
9.     if v is UNKNOWN and d[v] > weight(u, v), then do
10.      begin
11.      d[v] = weight(u, v)  // update with shorter weight
12.      p[v] = u             // update v's parent as v
13.      end
14.    end
15.  end
```

- Example (NOTE: v0 is the source vertex, and d[i] for each vertex i is also indicated in its circle):

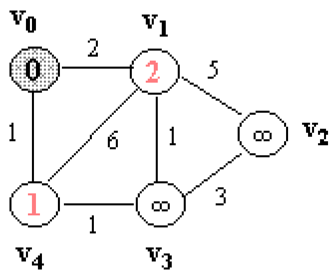
(0)



v	known	d_v	p_v
v_0	F	0	0
v_1	F	∞	0
v_2	F	∞	0
v_3	F	∞	0
v_4	F	∞	0

$Q = \{v_0, v_1, v_2, v_3, v_4\}$

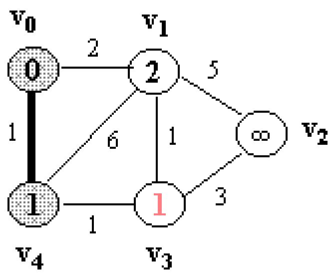
(1)



v	known	d_v	p_v
v_0	**T**	**0**	**0**
v_1	F	2	v_0
v_2	F	∞	0
v_3	F	∞	0
v_4	F	1	v_0

$Q = \{v_4, v_1, v_2, v_3\}$

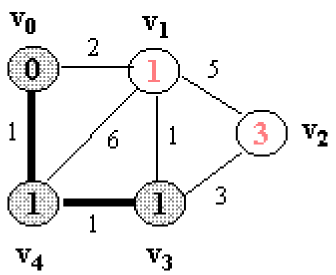
(2)



v	known	d_v	p_v
v_0	T	0	0
v_1	F	2	v_0
v_2	F	∞	0
v_3	F	1	v_4
v_4	**T**	**1**	$\mathbf{v_0}$

$Q = \{v_3, v_1, v_2\}$

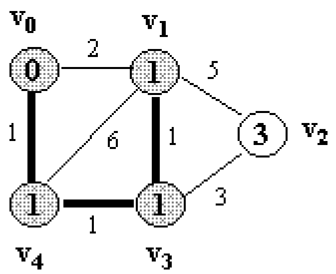
(3)



v	known	d_v	p_v
v_0	T	0	0
v_1	F	1	v_3
v_2	F	3	v_3
v_3	**T**	**1**	$\mathbf{v_4}$
v_4	T	1	v_0

$Q = \{v_1, v_2\}$

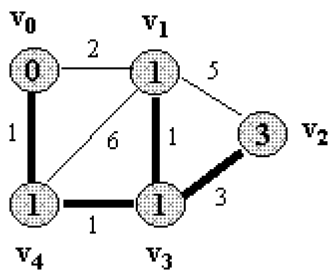
(4)



v	known	d_v	p_v
v_0	T	0	0
v_1	**T**	**1**	$\mathbf{v_3}$
v_2	F	3	v_3
v_3	T	1	v_4
v_4	T	1	v_0

$Q = \{v_2\}$

(5)



v	known	d_v	p_v
v_0	T	0	0
v_1	T	1	v_3
v_2	**T**	**3**	$\mathbf{v_3}$
v_3	T	1	v_4
v_4	T	1	v_0

$Q = \{\}$

6.2 Kruskal's Algorithm

- The main idea is to
  - start with a set (called **forest**) of singleton trees, and
  - **merge two trees at a time,** unless it creates a cycle in the merged tree, until the forest becomes one tree.
- The algorithm makes use of notions such as forest and **union-find** algorithm. But even without knowing them, you can intuitively understand Kruskal's algorithm quite easily.
- Algorithm:

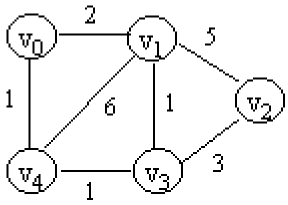
  Let G = (V, E) which is represented by an adjacency list **Adj**. Some support data structures:

  - **F** is the forest -- a set of all (partial) trees.
  - **MST** is the minimum spanning tree, represented by a **set of edges**.
  - **Q** is a **priority queue** of edges.

```
KRUSKAL(G)
1. Let F be a set of singleton set of all vertices in G.
2. MST <- {}
3. Q <- E
4. while Q not empty do
5.   (u, v) <- ExtractMin(Q)            // Q is modified
6.   if FIND-SET(u) != FIND-SET(v) then // FIND-SET(i) returns the set in F
                                        // which vertex i belongs to.
                                        // This effectively does cycle check.
                                        // If ACCEPTED,
7.     begin
8.     merge(FIND-SET(u),FIND-SET(v)) in F
9.     MST <- MST Union {(u, v)}
10.    end
11. return MST.
```

NOTE: In the figure below, a number in a vertex indicates the vertex number (NOT any kind of value).

(0)

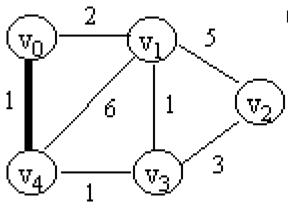


$F = \{\{v_0\}, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}\}$

$MST = \{\}$

$Q = \{(v_0,v_4), (v_1,v_3), (v_3,v_4),$
$(v_0,v_1), (v_2,v_3), (v_1,v_2), (v_1,v_4)\}$

(1)



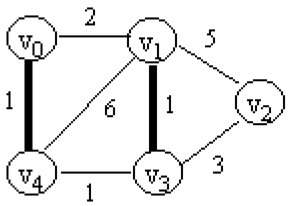
$(v_0,v_4) = \text{ExtractMin}(Q)$ -- ACCEPTED

$F = \{\{v_0, v_4\}, \{v_1\}, \{v_2\}, \{v_3\}\}$

$MST = \{(v_0,v_4)\}$

$Q = \{(v_1,v_3), (v_3,v_4),$
$(v_0,v_1), (v_2,v_3), (v_1,v_2), (v_1,v_4)\}$

(2)



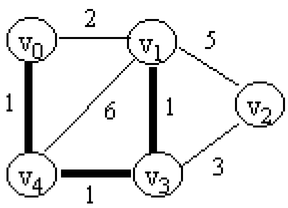
$(v_1,v_3) = \text{ExtractMin}(Q)$ -- ACCEPTED

$F = \{\{v_0, v_4\}, \{v_1, v_3\}, \{v_2\}\}$

$MST = \{(v_0,v_4), (v_1, v_3)\}$

$Q = \{(v_3,v_4),$
$(v_0,v_1), (v_2,v_3), (v_1,v_2), (v_1,v_4)\}$

(3)



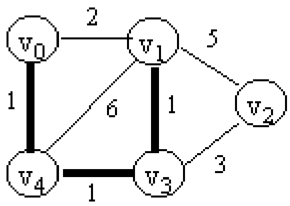
$(v_3,v_4) = \text{ExtractMin}(Q)$ -- ACCEPTED

$F = \{\{v_0, v_1, v_3, v_4\}, \{v_2\}\}$

$MST = \{(v_0,v_4), (v_1, v_3), (v_3,v_4)\}$

$Q = \{(v_0,v_1), (v_2,v_3), (v_1,v_2), (v_1,v_4)\}$

(4)



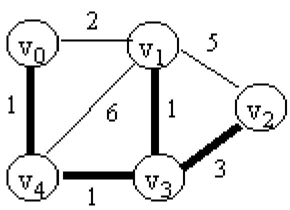
$(v_0,v_1) = \text{ExtractMin}(Q)$ -- REJECTED

$F = \{\{v_0, v_1, v_3, v_4\}, \{v_2\}\}$

$MST = \{(v_0,v_4), (v_1, v_3), (v_3,v_4)\}$

$Q = \{(v_2,v_3), (v_1,v_2), (v_1,v_4)\}$

(5)



$(v_2,v_3) = \text{ExtractMin}(Q)$ -- ACCEPTED

$F = \{\{v_0, v_1, v_2, v_3, v_4\}\}$

$MST = \{(v_0,v_4), (v_1, v_3), (v_3,v_4), (v_2,v_3)\}$

$Q = \{(v_1,v_2), (v_1,v_4)\}$

(6)

The algorithm continues until Q becomes empty, but since the forest has become one tree, all remaining edges in Q will be rejected and no no change will happen to MST.